Cauchy-Schwarz Inequality: Yet Another Proof

Titu Andreescu and Bogdan Enescu give an elegant — and memorable — proof of the Cauchy-Schwarz inequality among the gems in their *Mathematical Olympiad Treasures* (Birkhauser, 2003). Nominally, the proof is inductive, but what I like so much about it is that the induction step comes as close to being "computation free" as one can imagine.

The proof begins with a simple lemma that is useful in its own right.

LEMMA: For real a and b and for $x > 0$, and $y > 0$, one has

$$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}. \tag{1}$$

Naturally, this lemma is trivial — once it is conceived. Just by expansion and factorization one finds that it is simply a restatement of $(ay - bx)^2 \geq 0$. What I find attractive about this inequality is the way that it is *self-generalizing*. For example, if we replace b by $b + c$ and replace y by $y + z$ the we find

$$\frac{(a+b+c)^2}{x+y+z} \leq \frac{a^2}{x} + \frac{(b+c)^2}{y+z} \leq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}. \tag{2}$$

Repeating this step then gives us the general relationship

$$\frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + x_n} \leq \frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n}, \tag{3}$$

from which we can see Cauchy-Schwarz inequality at a glance. One simply sets

$$a_k = \alpha_k \beta_k \quad \text{and} \quad x_k = \beta_k^2.$$

How Different Really?

The related positivity bound $(a - bx)^2 \geq 0$ is a familiar starting place for proofs of the Cauchy-Schwarz inequality, so one can ask if this proof is really so different. The answer depends on where an individual chooses to draw some mental boundaries. To me, the appearance of the fractional bound (1) certainly makes the proof "new enough." Also, as I mentioned above, I find the induction here to be exceptionally cool.

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